A generalization of the Schur-Siegel-Smyth trace problem

Kyle Pratt
Department of Mathematics
The University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801
kpratt4@illinois.edu

George Shakan
Department of Mathematics
The University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801
shakan2@illinois.edu

Alexandru Zaharescu
Simion Stoilow Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, RO-014700 Bucharest, Romania
and
Department of Mathematics
The University of Illinois at Urbana-Champaign
1409 West Green Street, Urbana, IL 61801
zaharesc@illinois.edu

Abstract

Let α be a totally positive algebraic integer, and define its absolute trace to be $\frac{Tr(\alpha)}{\deg(\alpha)}$, the trace of α divided by the degree of α. Elementary considerations show that the absolute trace is always at least one, while it is plausible that for any $\epsilon > 0$, the absolute trace is at least $2 - \epsilon$ with only finitely many exceptions. This is known as the Schur-Siegel-Smyth trace problem. Our aim in this paper is to show that the Schur-Siegel-Smyth trace problem can be considered as a special case of a more general problem.

Keywords and Phrases: Minimal Polynomial, Totally Positive Algebraic Number, Schur-Siegel-Smyth trace problem

2010 Mathematics Subject Classification 11B83, 11R99

1 Introduction

Let $\mathcal{F}$ be the set of polynomials $f(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0 \in \mathbb{Z}[x]$ with f monic, irreducible, and having only positive real roots. If α is the root of such a polynomial then we say that α is a *totally positive algebraic integer*. We define the *absolute trace* $A(f) := \frac{a_{n-1}}{n}$, and identify the absolute trace of α with the absolute trace of its minimal polynomial. It is interesting to investigate how small $A(f)$ can be as f ranges through $\mathcal{F}$.

An elementary starting point comes from the arithmetic-geometric mean inequality. For positive real numbers $x_1, \ldots, x_n$, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}. \tag{1}$$

If the x_i are the roots of a polynomial $f \in \mathcal{F}$, then $x_1 \cdots x_n = a_0$ is a positive integer, and therefore $x_1 \cdots x_n \geq 1$. We conclude that $A(f) \geq 1$ for all $f \in \mathcal{F}$. Since the x_i are the roots of some $f \in \mathcal{F}$, we expect that (1) is far from optimal for $n > 1$. For instance, the discriminant,

$$\Delta := \prod_{1 \leq i < j \leq n} (x_i - x_j)^2,$$

is at least one, which is not true for n arbitrary positive real numbers.

The main problem is to find the supremum of all positive ρ such that for any $\epsilon > 0$, there exist only finitely many $f \in \mathcal{F}$ with $A(f) \leq \rho - \epsilon$. An elegant construction due to Siegel [9] shows that $\rho \leq 2$, as follows. Let p be an odd prime, and let ζ be a primitive pth root of unity. Consider the polynomial

$$g(x) := \prod_{i=1}^{\frac{p-1}{2}} (x - (\zeta^i + \zeta^{-i} + 2)).$$

Note that $\zeta^i + 1, \zeta^{-i} + 1$ are nonzero and lie on the circle $\{z : |z - 1| = 1\}$, so all the roots of g are positive. One may easily check that $A(g) = 2 - \frac{2}{p-1}$. Siegel's observation gives rise to the following problem (P17 of [3]; see also Problem 1 of [5]).

Problem 1.1 (Schur-Siegel-Smyth trace problem). *For every $\epsilon > 0$, there are only finitely many $f \in \mathcal{F}$ with $A(f) \leq 2 - \epsilon$.*

There has been partial progress towards Problem 1.1. Schur [8] improved (1) and showed that $\rho \geq e^{1/2} = 1.6487\ldots$. Siegel [9] further improved (1) and obtained $\rho \geq 1.7336\ldots > e^{11/20}$. Both Schur and Siegel used arguments based on properties of the discriminant. Using an argument based on resultants, Smyth [10, 11] showed that $\rho \geq 1.7719$. Smyth [12] and Serre (Appendix B of [2]) showed this technique will not allow one to resolve Problem 1.1. Smyth [12] suggests that perhaps $\rho < 2$.

Other work on Problem 1.1 has been done by Aguirre and Peral [2], Flammang [4], Stan and the third author [14], Liang and Wu [5], and McKee [6].

In this paper we propose a generalization of Problem 1.1, in which we consider the joint behavior of all the coefficients of $f \in \mathcal{F}$. To motivate the conjecture give some notation and a lemma.

For $f(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0 \in \mathcal{F}$ Viete's relations imply $a_j > 0$ for $j = 0, \ldots, n-1$. The Maclaurin inequalities state that

$$\frac{a_{n-1}}{n} = \frac{a_{n-1}}{\binom{n}{1}} \geq \left(\frac{a_{n-2}}{\binom{n}{2}}\right)^{1/2} \geq \cdots \geq \left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \geq \cdots \geq \left(\frac{a_0}{\binom{n}{n}}\right)^{1/n} = a_0^{1/n}. \tag{2}$$

We aim to understand the behavior of all the coefficients simultaneously, so we consider

$$\mathcal{P} := \left\{ \left(d/n, \left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \right) \in \mathbb{R}^2 : f \in \mathcal{F},\ 1 \leq d \leq n \right\}.$$

We find that the points of $\mathcal{P}$ coming from Siegel's construction above tend to a limiting curve. The following lemma, which we prove in a later section, allows us to explicitly calculate this curve.

Lemma 1.2. *Let p be an odd prime and let ζ be a primitive pth root of unity. Let $g(x)$ be the minimal polynomial of $\zeta + \zeta^{-1} + 2$. Then*

$$g(x) = \sum_{d=0}^{n} (-1)^{n-d} \binom{n+d}{2d} x^d,$$

where $n = \frac{p-1}{2}$.

From Lemma 1.2 we formulate a conjecture. Let $d = cn + O(1)$ (for example, $d = \lfloor cn \rfloor$), where $c \in (0,1)$. For $0 < b < a$, define $h(a,b) := a \log a - b \log b - (a-b)\log(a-b)$. Using Stirling's formula we find that

$$\left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \longrightarrow L(c) := e^{\frac{h(2-c,2-2c)-h(1,c)}{c}}$$

as $n \to \infty$.

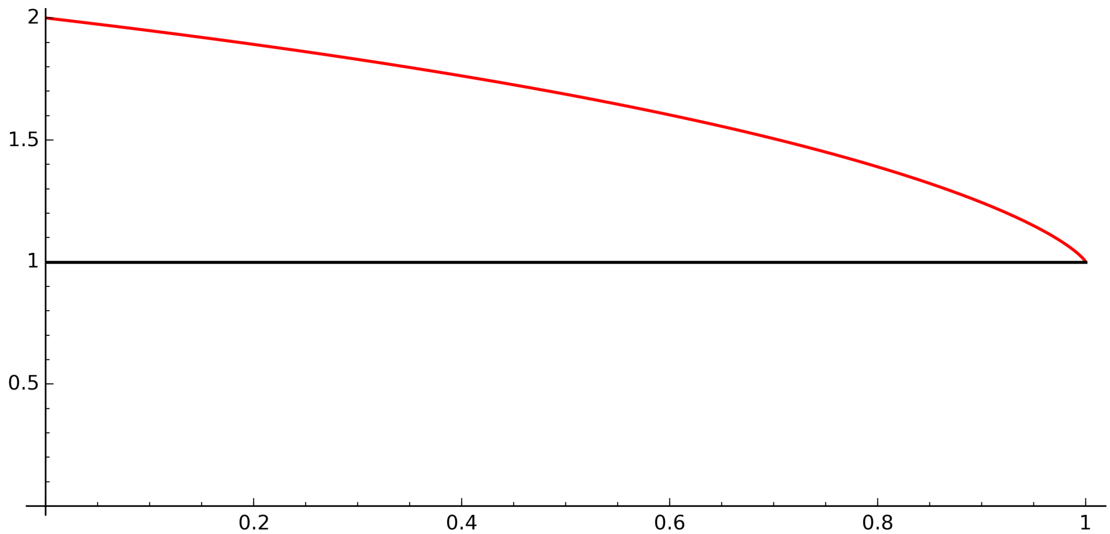
Conjecture 1.3. *Fix $\epsilon > 0$. Then there exist only finitely many $f(x) = \sum_{i=0}^{n} (-1)^{n-i} a_i x^i \in \mathcal{F}$ such that*

$$\left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \leq L(d/n) - \epsilon$$

for some d with $0 \leq d \leq n-1$.

Observe that Conjecture 1.3 resolves Problem 1.1. Figure 1 shows a graph of $L(c)$. The horizontal line $y = 1$ represents the trivial lower bound on $\left(a_{n-d}/\binom{n}{d}\right)^{1/d}$ derived from (2). Actually, Pritsker (Problem B, [7]) was the first to propose the study of the limit points of $a_{n-d}/\binom{n}{d}$ in relation to the Schur-Siegel-Smyth trace problem. He studied the case where d is fixed and n tends to infinity, while in this note we are interested in the simultaneous behavior of all the coefficients.

Figure 1: Graph of curve from Conjecture 1.3



We support Conjecture 1.3 with a lower bound; we describe our approach, which generalizes the method of Siegel [9]. Let $x_1, \ldots, x_n$ be distinct positive real numbers, and let $f(x) := \prod_{i=1}^{n}(x - x_i)$. Define a_k for $k = 0, \ldots, n-1$ via $f(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$, where $a_j > 0$. From (2) we obtain

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{1/(n-k)} \geq a_0^{1/n}. \tag{3}$$

We prove a generalization of (3). Let $\Delta := \prod_{1 \leq i < j \leq n}(x_i - x_j)^2$, and define functions

$$P(t) := \frac{1}{n!}\prod_{j=0}^{n-2}\left(\frac{t+j}{n-j}\right)^{n-j-1},$$

$$Q_k(t) := \frac{(t+k)^k(t+k+1)^k \cdots (t+n-1)^k}{t^{n-k}(t+1)^{n-k} \cdots (t+k-1)^{n-k}}.$$

We have the following theorem.

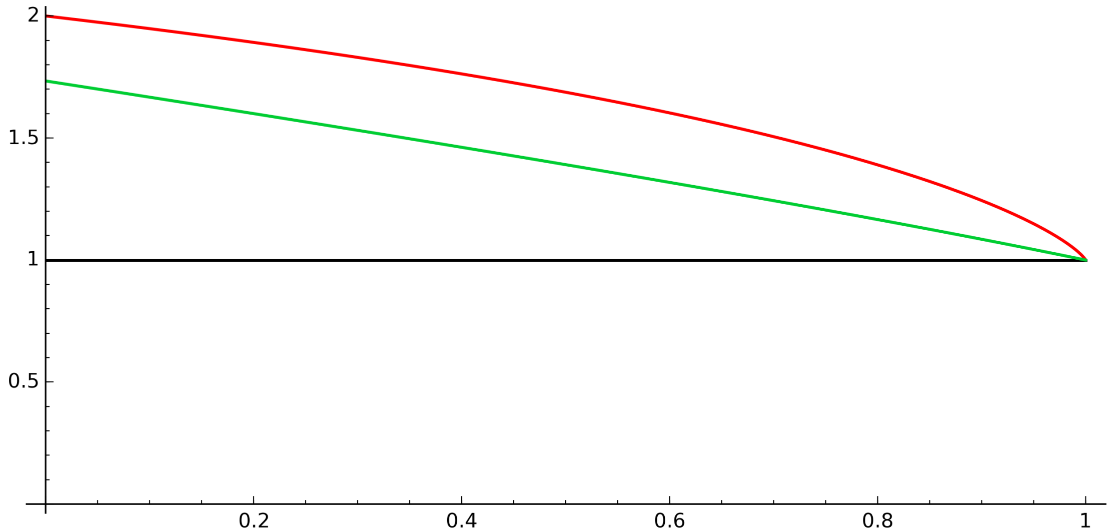
Theorem 1.4. *Let $x_1, \ldots, x_n$ be distinct positive real numbers. Let μ_0 be the positive root of $P(t) = \frac{a_0^{n-1}}{\Delta}$. Then for $k = 0, \ldots, n-1$ we have*

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{n} \geq Q_k(\mu_0)a_0^{n-k}.$$

When $k = n-1$ this is precisely Siegel's Theorem 1 of [9], and we generalize the argument given there. As the coefficients of $P(t)$ are positive and $P(0) = 0$ we see that μ_0 exists and is uniquely determined. Observe that for $0 \leq a < b$ we have

$$\frac{t+b}{t+a} = 1 + \frac{b-a}{t+a} > 1,$$

Figure 2: Graph of curve from Theorem 1.5 compared with curve from Conjecture 1.3



so $Q_k(t) > 1$ for $t > 0$. Thus Theorem 1.4 is an improvement of (3).

We apply Theorem 1.4 to make progress towards Conjecture 1.3.

Theorem 1.5. *Let ϑ be the unique positive solution to the transcendental equation*

$$(1+\vartheta)^2 \log(1+\vartheta^{-1}) + \log\vartheta - \vartheta - 1 = 0.$$

Define a curve $\ell(c) : [0,1] \to \mathbb{R}$ by

$$\ell(c) := exp\left(\frac{(1-c)(\vartheta+1)\log(\vartheta+1) + c\vartheta\log(\vartheta) + (c-1-\vartheta)\log(\vartheta+1-c)}{c}\right)$$

for $c > 0$, and $\ell(0)$ defined by continuity. For all $\epsilon > 0$, there are only finitely many $f \in \mathcal{F}$ such that

$$\left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \leq \ell(d/n) - \epsilon$$

for some $d \in \{1, \ldots, n-1\}$.

A graph of the curve in Theorem 1.5 appears as the middle curve in Figure 2. The upper curve is the curve in Conjecture 1.3, and the horizontal line $y = 1$ represents the trivial lower bound.

Using a numerical integration package one finds that the area of the region bounded by the curves in Figure 1 is $\approx .63917$. The area bounded by the horizontal line and the middle curve in Figure 2 is $\approx .38323$. Theorem 1.5 therefore shows that $\approx 59.95\%$ of the region in Figure 1 cannot contain limit points of $\mathcal{P}$. It is likely that Theorem 1.5 could be improved by generalizing the methods of Smyth [10, 11] and his successors (e.g. [2, 4, 5, 6]).

A natural problem that arises is to determine the set of limit points of $\mathcal{P}$ that lie above the curve of Conjecture 1.3. One result in this direction is due to Smyth [11], who showed that every point $(0, x)$ with $x \geq 2$ is a limit point of $\mathcal{P}$.

2 Proof of Lemma 1.2

We make use of the Chebyshev polynomials T_m and U_m, the mth Chebyshev polynomials of the first and second kind, respectively.

Recall that for $m \in \mathbb{N}$, the Chebyshev polynomial of the first kind, $T_m \in \mathbb{Q}[x]$, is defined by

$$T_m(\cos\theta) = \cos(m\theta)$$

while the Chebyshev polynomial of the second kind is defined by

$$U_m(\cos\theta) = \frac{\sin((m+1)\theta)}{\sin\theta}.$$

We make use of the identity

$$U_m(x/2) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m-k}{k} x^{m-2k} = \sum_{k=0}^{\infty} (-1)^k \binom{m-k}{k} x^{m-2k}. \tag{4}$$

The first equality follows from (1.15) and (2.32) of [1], and the second equality follows from the first since $\binom{m-k}{k} = 0$ for $k > \lfloor m/2 \rfloor$.

Lemma 2.1. *Let p be an odd prime, and let ζ be a primitive pth root of unity. Let $f(x)$ be the minimal polynomial of $\zeta + \zeta^{-1}$. Then $f(x) = U_n(x/2) + U_{n-1}(x/2)$, where $n := \frac{p-1}{2}$.*

Proof. We include the proof for completeness, since it does not seem to appear in the literature. The argument we present is due to Zieve [15].

Let p be an odd prime, and let ζ be a primitive pth root of unity. Define $\omega := \zeta + \zeta^{-1}$ and $n := \frac{p-1}{2}$. The degree of the minimal polynomial of ω is n, so it suffices to find a monic polynomial of degree n of which ω is a root.

Write $z = e^{i\theta} = \cos\theta + i\sin\theta$, so that $z + z^{-1} = 2\cos\theta$ and therefore $T_p(\frac{z+z^{-1}}{2}) = \frac{z^p+z^{-p}}{2}$. It follows that $T_p(\frac{Z+Z^{-1}}{2}) = \frac{Z^p+Z^{-p}}{2}$ as rational functions in Z, since equality holds for infinitely many values of Z. This implies $T_p(\frac{\omega}{2}) = \frac{\zeta^p+\zeta^{-p}}{2} = 1$, so ω is a root of $T_p(\frac{x}{2}) - 1$.

Change variables to $X = Z + Z^{-1}$, so that

$$T_p(X/2) - 1 = \frac{Z^p + Z^{-p}}{2} - 1 = \frac{1}{2}(Z^{p/2} - Z^{-p/2})^2 = \frac{X-2}{2}\left(\frac{Z^{p/2} - Z^{-p/2}}{Z^{1/2} - Z^{-1/2}}\right)^2.$$

We show that $\frac{Z^{p/2}-Z^{-p/2}}{Z^{1/2}-Z^{-1/2}} = U_n(X/2) + U_{n-1}(X/2)$, following Zieve [16]. Multiplying numerator and denominator by $Z^{1/2} + Z^{-1/2}$, we obtain

$$\frac{Z^{p/2} - Z^{-p/2}}{Z^{1/2} - Z^{-1/2}} = \frac{Z^{\frac{p-1}{2}+1} - Z^{-\left(\frac{p-1}{2}+1\right)}}{Z - Z^{-1}} + \frac{Z^{\frac{p-3}{2}+1} - Z^{-\left(\frac{p-3}{2}+1\right)}}{Z - Z^{-1}}.$$

We also have that

$$U_m\left(\frac{Z+Z^{-1}}{2}\right) = \frac{Z^{m+1} - Z^{-(m+1)}}{Z - Z^{-1}}.$$

We deduce that ω is a root of

$$T_p(X/2) - 1 = \frac{X-2}{2}\left(U_n(X/2) + U_{n-1}(X/2)\right)^2.$$

Since ω is not a root of $X-2$ we must have that ω is a root of $f(x) := U_n(X/2) + U_{n-1}(X/2)$. However, one easily checks that f is monic and has degree n, so that $f(x)$ must be the minimal polynomial of ω. $\square$

Let p be an odd prime and let ζ be a primitive pth root of unity. We wish to determine the minimal polynomial $g(x)$ of $\zeta + \zeta^{-1} + 2$, which we determine from the polynomial $f(x)$ of Lemma 2.1. Indeed, elementary Galois theory implies that $g(x) = f(x-2)$, so the two polynomials are closely related. Recalling Lemma 2.1 and equation (4), we determine the coefficients of

$$\sum_{k=0}^{\infty} \binom{m-k}{k} (-1)^k (x-2)^{m-2k}$$

for all m.

Our approach is via generating functions, and we work both formally and analytically. We define $A(u,v) := \sum_{m,j\geq 0} a_{m,j} u^m v^j$, where $a_{m,j}$ is the coefficient of x^j in $\sum_{k=0}^{\infty} \binom{m-k}{k}(-1)^k(x-2)^{m-2k}$. We first find $A(u,v)$ in closed form.

Lemma 2.2. *With $A(u,v)$ and $a_{m,j}$ as above, we have $A(u,v) = \frac{1}{1+2u-uv+u^2}$.*

Proof. Expanding out with the binomial theorem and extending the sum to infinity as before, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{m-k}{k} (-1)^k (x-2)^{m-2k} &= \sum_{k=0}^{\infty} \binom{m-k}{k} (-1)^k \sum_{j=0}^{\infty} \binom{m-2k}{j} x^j (-2)^{m-2k-j} \\ &= \sum_{j=0}^{\infty} x^j (-2)^{m-j} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{m-k}{k}\binom{m-2k}{j}}{4^k}.\end{aligned}$$

We ignore the factor of $(-2)^{m-j}$ for the moment. Set $b_{m,j} := \sum_{k=0}^{\infty} \frac{(-1)^k\binom{m-k}{k}\binom{m-2k}{j}}{4^k}$, and define $B(u,v) := \sum_{m,j\geq 0} b_{m,j} u^m v^j$. We obtain

$$\begin{aligned}B(u,v) &= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\binom{m-k}{k}\binom{m-2k}{j}}{4^k} u^m v^j = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \sum_{m=0}^{\infty} \binom{m-k}{k} u^m \sum_{j=0}^{\infty} \binom{m-2k}{j} v^j \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \sum_{m=0}^{\infty} \binom{m-k}{k} u^m (1+v)^{m-2k}.\end{aligned}$$

Changing variables to $r = m-2k$, this yields

$$B(u,v) = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{4^k} \sum_{r=0}^{\infty} \binom{r+k}{k} (u(1+v))^r.$$

By the elementary identity

$$\frac{1}{(1-y)^N} = \sum_{i=0}^{\infty} \binom{i+N-1}{i} y^i$$

we find that

$$B(u,v) = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{4^k} \frac{1}{(1-u(1+v))^{k+1}}.$$

Summing the geometric series and simplifying, we obtain

$$B(u,v) = \frac{4}{4-4u-4uv+u^2} = \frac{1}{1-u-uv+\left(\frac{u}{2}\right)^2}.$$

We remark that all of these formal manipulations are analytically rigorous for $|u|, |v|$ sufficiently small.

From the relation $a_{m,j} = (-2)^{m-j} b_{m,j}$, we see that replacing u by $-2u$ and v by $-\frac{v}{2}$ gives

$$A(u,v) = B(-2u, -v/2) = \frac{1}{1+2u-uv+u^2}.$$

$\square$

Proof of Lemma 1.2. The coefficient of x^j in $g(x)$ is equal to $a_{n,j} + a_{n-1,j}$. It follows by Lemma 2.2 that a generating function for the coefficients of $g(x)$ is given by

$$G(u,v) := \frac{1+u}{1+2u-uv+u^2} = \sum_{m,j \geq 0} g_{m,j} u^m v^j.$$

We find $g_{m,j}$ using complex analysis. Applying Cauchy's integral formula twice, we have

$$g_{m,j} = \frac{1}{2\pi i} \int_{C_1} \frac{1}{u^{m+1}} \left(\frac{1}{2\pi i} \int_{C_2} \frac{G(u,v)}{v^{j+1}} dv \right) du,$$

where C_1, C_2 are circles centered at the origin of sufficiently small radius.

For fixed u, note that $G(u,v)$ has a pole at $v = \frac{(1+u)^2}{u}$. Let γ be a circle of sufficiently small radius about $\frac{(u+1)^2}{u}$. By considering a circular contour of sufficiently large radius containing both C_2 and γ and applying the residue theorem, we see that

$$\frac{1}{2\pi i} \int_{C_2} \frac{G(u,v)}{v^{j+1}} dv = -\frac{1}{2\pi i} \int_{\gamma} \frac{G(u,v)}{v^{j+1}} dv.$$

Applying the residue theorem again to evaluate the latter integral yields

$$-\frac{1}{2\pi i} \int_{\gamma} \frac{G(u,v)}{v^{j+1}} dv = \lim_{v \to \frac{(u+1)^2}{u}} \frac{1+u}{u} \frac{1}{v^{j+1}} = \frac{u^j}{(1+u)^{2j+1}},$$

so we have

$$g_{m,j} = \frac{1}{2\pi i}\int_{C_1} \frac{1}{u^{m-j+1}(1+u)^{2j+1}}du.$$

Letting Γ be a circle centered at -1 of sufficiently small radius and arguing as before we have

$$\frac{1}{2\pi i}\int_{C_1} \frac{1}{u^{m-j+1}(1+u)^{2j+1}}du = -\frac{1}{2\pi i}\int_{\Gamma} \frac{1}{u^{m-j+1}(1+u)^{2j+1}}du.$$

It remains to determine the residue of the integrand at $u = -1$ and apply the residue theorem. Expanding $\frac{-1}{u^{m-j+1}}$ in powers of $1+u$ gives

$$\frac{-1}{u^{m-j+1}} = \frac{(-1)^{m-j}}{(1-(1+u))^{m-j+1}} = (-1)^{m-j}\sum_{n=0}^{\infty}\binom{n+m-j}{n}(1+u)^n.$$

The residue comes from the term with $n = 2j$, which yields

$$g_{m,j} = (-1)^{m-j}\binom{m+j}{2j}.$$

This completes the proof. $\square$

We remark that the denominator of $G(u,v)$ being linear in v greatly simplifies the proof of Lemma 1.2.

3 Proof of Theorem 1.4

Before embarking on the proof, we first give some idea of the nature of these inequalities. With the notation as before, define $S_k := \frac{a_{n-k}}{\binom{n}{k}}$. Newton's inequalities give $S_k^2 \geq S_{k-1}S_{k+1}$ for $k = 1,\ldots,n-1$. From Siegel's proof of Theorem 1 in [9] one can deduce that $S_k^2 \geq S_{k-1}S_{k+1}\left(1+\frac{1}{\mu_0+k-1}\right)$, where μ_0 is defined as above. As we are interested in the simultaneous behavior of the coefficients, it is more convenient for us to prove Theorem 1.4.

Proof of Theorem 1.4. We follow the method of Siegel [9]. Let a_{n-1}, a_0 be positive real numbers such that $(\frac{a_{n-1}}{n})^n > a_0$. Consider $y_1,\ldots,y_n \in \mathbb{R}$ subject to

- $y_j \geq 0$ for $j = 1,\ldots,n$,
- $y_1 + \cdots + y_n = a_{n-1}$,
- $y_1 \cdots y_n = a_0$.

Siegel (section 2 of [9]) maximized $\Delta(\boldsymbol{y}) := \prod_{1\leq i<j\leq n}(y_i - y_j)^2$ subject to these constraints. To this end, he used Lagrange multipliers on the function $\phi(y_1,\ldots,y_n) = \frac{1}{2}\log\Delta(\boldsymbol{y}) - \lambda(y_1+\cdots+y_n) + \mu\log(y_1\cdots y_n)$. He found the relation

$$P(\mu) = \frac{a_0^{n-1}}{\max_{\boldsymbol{y}}\Delta(\boldsymbol{y})}. \tag{5}$$

Taking $x_1, \ldots, x_n$ to be distinct positive real numbers with $x_1 + \cdots + x_n = a_{n-1}, x_1 \cdots x_n = a_0$, set $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)^2$, so that (5) gives

$$P(\mu) = \frac{a_0^{n-1}}{\max_{\boldsymbol{y}} \Delta(\boldsymbol{y})} \le \frac{a_0^{n-1}}{\Delta}.$$

Since $P(t)$ is monotone increasing, $P(0) = 0$, and $\lim_{t \to \infty} P(t) = \infty$, there exists a unique $\mu_0 \ge \mu$ such that $P(\mu_0) = \frac{a_0^{n-1}}{\Delta}$. Writing (10) of [9] in our notation, we have

$$(-1)^k S_k = (\mu + n - k)(\mu + n - k + 1) \cdots (\mu + n - 1) \lambda^{-k},$$

which implies

$$\frac{S_k^n}{a_0^{n-k}} = Q_k(\mu) \ge Q_k(\mu_0).$$

For the inequality we have used the fact that $Q_k(t)$ is monotone decreasing for $t > 0$. $\square$

4 Proof of Theorem 1.5

We begin with a lemma.

Lemma 4.1. *Let $n \ge 2$ be an integer, and let $0 \le k \le n-1$. Then the function $Q_k(t)^{n-1} P(t)^{n-k}$ is monotone increasing for $t > 0$.*

Proof. It suffices to show that $\frac{d}{dt} \log(Q_k(t)^{n-1} P(t)^{n-k}) > 0$. We take n to be fixed, and proceed by backwards induction on k, with $k = n - 1$ being the base case. Taking logarithms and derivatives, we have

$$\begin{aligned} \frac{d}{dt} \log(Q_k(t)^{n-1} P(t)^{n-k}) &= k(n-1) \sum_{j=k}^{n-1} \frac{1}{t+j} - (n-1)(n-k) \sum_{j=0}^{k-1} \frac{1}{t+j} \\ &\quad + (n-k) \sum_{j=0}^{n-2} \frac{n-j-1}{t+j} \\ &= k(n-1) \sum_{j=k}^{n-1} \frac{1}{t+j} + (n-1)(n-k) \sum_{j=k}^{n-2} \frac{1}{t+j} - (n-k) \sum_{j=0}^{n-2} \frac{j}{t+j}. \end{aligned} \tag{6}$$

When $k = n - 1$, we argue as in the proof of (5) in [9]. Here (6) becomes

$$\frac{(n-1)^2}{t+n-1} - \sum_{j=0}^{n-2} \frac{j}{t+j} = \sum_{j=0}^{n-2} \left(\frac{n-1}{t+n-1} - \frac{j}{t+j} \right) = \sum_{j=0}^{n-2} \frac{t(n-j-1)}{(t+n-1)(t+j)} > 0$$

for $t > 0$. This gives the base case.

Now suppose that (6) is positive for $1 \le k \le n-1$ and $t > 0$; we wish to show it is positive for $t > 0$ when k is replaced by $k-1$. Replacing k by $k-1$ in (6) and simplifying, we have that

$$\frac{d}{dt}\log(Q_{k-1}(t)^{n-1}P(t)^{n-(k-1)}) = \frac{d}{dt}\log(Q_k(t)^{n-1}P(t)^{n-k}) + \frac{n(n-1)}{t+k-1} - \sum_{j=1}^{n-1}\frac{j}{t+j}.$$

By the induction hypothesis, it suffices to show that $\frac{n(n-1)}{t+k-1} - \sum_{j=1}^{n-1}\frac{j}{t+j}$ is positive, which follows from

$$\frac{n(n-1)}{t+k-1} - \sum_{j=1}^{n-1}\frac{j}{t+j} = \sum_{j=1}^{n-1}\left(\frac{n}{t+k-1} - \frac{j}{t+j}\right) = \sum_{j=1}^{n-1}\frac{t(n-j)+j(n-k+1)}{(t+k-1)(t+j)} > 0$$

for $t > 0$ and $k \ge 1$. Thus (6) is positive when k is replaced by $k-1$, and the induction is complete. $\square$

Proof of Theorem 1.5. Here we follow the proof of Theorem 2 in [9]. Let $f(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0 \in \mathcal{F}$. From Theorem 1.4 we have

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{n(n-1)} \ge Q_k(\mu)^{n-1}a_0^{(n-1)(n-k)} \ge Q_k(\mu)^{n-1},$$

since a_0 is a positive integer and therefore $a_0 \ge 1$. On the other hand, the relation $P(\mu) = \frac{a_0^{n-1}}{\Delta}$ gives

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{n(n-1)} \ge Q_k(\mu)^{n-1}\Delta^{n-k}P(\mu)^{n-k},$$

which implies

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{n(n-1)} \ge \max\left(Q_k(\mu)^{n-1}, \Delta^{n-k}Q_k(\mu)^{n-1}P(\mu)^{n-k}\right).$$

Recall that $Q_k(t)^{n-1}$ is monotone decreasing and Lemma 4.1 shows $Q_k(\mu)^{n-1}P(\mu)^{n-k}$ is monotone increasing. Thus if $t > \mu$ we have $Q_k(t) \le Q_k(\mu)$, and if $0 < t \le \mu$ we have $Q_k(t)^{n-1}P(t)^{n-k} \le Q_k(\mu)^{n-1}P(\mu)^{n-k}$. It follows that

$$\begin{aligned}\max\left(Q_k(\mu)^{n-1}, \Delta^{n-k}Q_k(\mu)^{n-1}P(\mu)^{n-k}\right) &\ge \min(Q_k(t)^{n-1}, \Delta^{n-k}Q_k(t)^{n-1}P(t)^{n-k}) \\ &= Q_k(t)^{n-1}\min(1, \Delta^{n-k}P(t)^{n-k}).\end{aligned}$$

Thus, if t is a positive number such that $\Delta P(t) \ge 1$, we have

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{n(n-1)} \ge Q_k(t)^{n-1}. \tag{7}$$

Let $g(v) = (1+v)^2 \log(1+v^{-1}) + \log v - v - 1$. One may check that $g'(v) > 0$ for $v > 0$, $\lim_{v\to 0^+} g(v) = -1$, and $\lim_{v\to\infty} g(v) = \infty$, so that the equation $g(v) = 0$ has exactly one positive root $\vartheta = .3144808\ldots$. Observe that $g(v) > 0$ if $v > \vartheta$. Siegel showed (see (16) of [9] and surrounding discussion) that if $v > \vartheta$ then $P(vn) \to \infty$ as $n \to \infty$. Thus, since $\Delta \geq 1$ we conclude by (7) that $\left(a_k/\binom{n}{k}\right)^{n(n-1)} \geq Q_k(vn)^{n-1}$ for n sufficiently large.

Let $t = vn$ and $k = cn + O(1)$ for $v > \vartheta$ and $c \in (0,1)$. By Euler-Maclaurin summation we find that

$$\begin{aligned}
\log Q_k(t) &= k\sum_{j=k}^{n-1} \log(t+k) - (n-k)\sum_{j=0}^{k-1} \log(t+j) \\
&= k[n(1+v)\log((1+v)n) - (1+v)n] + k[-vn\log n + vn] \\
&+ n[-(k+vn)\log(k+vn) + k + vn + vn\log(vn) - vn] + O(n\log n) \\
&= n^2(c(1+v)\log(1+v) - cv\log v + v\log v - (c+v)\log(c+v)) + O(n\log n).
\end{aligned}$$

Dividing by $(1-c)n^2$ and exponentiating, we find that

$$\left(\frac{a_k}{\binom{n}{k}}\right)^{\frac{1}{n-k}} \geq K_v(c) e^{O\left(\frac{\log n}{(1-c)n}\right)} = K_v(c) + O\left(\frac{\log n}{(1-c)n}\right), \tag{8}$$

where

$$K_v(c) := \exp\left(\frac{c(v+1)\log(v+1) + (1-c)v\log v - (c+v)\log(v+c)}{1-c}\right).$$

Now, let $k = cn + O(1)$. In order to orient the curve from left to right, we let $\ell_v(c) = K_v(1-c)$. Fixing $\epsilon, \delta > 0$ we use that the error term in (8) is uniform for $c \geq \delta$ to conclude that

$$\left(\frac{a_{n-k}}{\binom{n}{k}}\right)^{1/k} \geq \ell_v(c) - \epsilon,$$

for all $c \geq \delta$ and for all n large enough. By the Maclaurin inequalities and the fact that $\ell_v(c)$ is monotone decreasing, it follows that

$$\left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \geq \ell_v(d/n) - \epsilon - (\ell_v(0) - \ell(\delta)),$$

for all $d \in \{1,\ldots,n\}$, when n is large enough. By the continuity of $\ell_v(c)$ at $c = 0$, we obtain that for all $\epsilon' > 0$, n large, and $d \in \{1,\ldots,n\}$,

$$\left(\frac{a_{n-d}}{\binom{n}{d}}\right)^{1/d} \geq \ell_v(d/n) - \epsilon'.$$

To conclude Theorem 1.5, it is enough to show we can replace $\ell_v(c)$ with $\ell_\vartheta(c) = \ell(c)$, which is permitted since $\ell_v(c)$, as a function of c and v, is continuous on a compact set and therefore uniformly continuous.

$\square$

Acknowledgements

The first author is supported by the National Science Foundation Graduate Research Fellowship Program under Grant Number DGE-1144245. The authors used SAGE [13] version 6.7 for computations. We thank Michael Zieve and the anonymous referee for helpful comments.

References

[1] S. Abhyankar, S. Cohen, M. Zieve. *Bivariate factorizations connecting Dickson polynomials and Galois theory*, Trans. Amer. Math. Soc. 352 (2000), no. 6, 28712887.

[2] J. Aguirre, J.C. Peral. *The trace problem for totally positive algebraic integers*, Number Theory and Polynomials. (Conference proceedings, University of Bristol, 3-7 April 2006, editors J.F. McKee and C.J. Smyth). LMS Lecture notes

[3] P. Borwein. *Computational excursions in analysis and number theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 10. Springer-Verlag, New York, 2002.

[4] V. Flammang. *Trace of totally positive algebraic integers and integer transfinite diameter*, Math. Comp. **78** (2009), no. 266, 1119-1125

[5] Y. Liang, Q. Wu. *The trace problem for totally positive algebraic integers*, J. Aust. Math. Soc. **90** (2011), no. 3, 341-354

[6] J. McKee. *Computing totally positive algebraic integers of small trace*, Math. Comp. **80** (2011), no. 274, 1041-1052

[7] I. Pritsker. *Asymptotic distribution and symmetric means of algebraic numbers*, Acta Arith. 168 (2015), no. 2, 121-138.

[8] I. Schur. *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Zeitschr. 1 (1918), no. 4, 377-402

[9] C. L. Siegel. *The trace of totally positive and real algebraic integers*, Ann. of Math. (2) **46**, (1945). 302-312

[10] C. J. Smyth. *Totally positive algebraic integers of small trace*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 3, 1-28

[11] C. J. Smyth. *The mean values of totally real algebraic integers*, Math. Comp. **42** (1984), no. 166, 663-681

[12] C. J. Smyth. *An inequality for polynomials*, Number theory (Ottawa, ON, 1996), 315321, CRM Proc. Lecture Notes, 19, Amer. Math. Soc., Providence, RI, 1999.

[13] W. A. Stein et al. Sage Mathematics Software (Version 6.7), The Sage Development Team, 2015, http://www.sagemath.org.

[14] F. Stan, A. Zaharescu. *Siegel's trace problem and character values of finite groups*, J. Reine Angew. Math. **637** (2009), 217-234.

[15] M. Zieve (http://math.stackexchange.com/users/61116/michael-zieve). *Minimal Polynomial of* $\zeta + \zeta^{-1}$, URL (version: 2015-06-10): http://math.stackexchange.com/questions/460930/minimal-polynomial-of-zeta-zeta-1

[16] M. Zieve (http://math.stackexchange.com/users/61116/michael-zieve). *Is it true that* $p|[\mathbb{Q}(\cos(\frac{\pi}{p^2}) : \mathbb{Q})]$*?*, URL (version: 2015-06-11): http://math.stackexchange.com/questions/464300/is-it-true-that-p-mid-mathbb-q-cos-frac-pip2-mathbb-q/464480#464480